Uncertainty relation of mixed states by means of Wigner-Yanase-Dyson information

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Abstract

The variance of an observable in a quantum state is usually used to describe Heisenberg uncertainty relation. For mixed states, the variance includes quantum uncertainty and classical uncertainty. By means of the skew information and the decomposition of the variance, a stronger uncertainty relation was presented by Luo in [Phys. Rev. A 72, 042110 (2005)]. In this paper, by using Wigner-Yanase-Dyson information which is a generalization of the skew information, we propose a general uncertainty relation of mixed states.

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1 Introduction

In quantum measurement theory, the Heisenberg uncertainty principle provides a fundamental limit for the measurements of incompatible observables. On the other hand, as dictated by Cramer-Rao's lower bound, there is also an ultimate limit for the resolution of any unbiased parameter (see for instance, [1]), and this lower bound is given by a quantity called Fisher information. A long time ago, Wigner demonstrated that it is more difficult to measure observables that do not commute with some additive conserved quantity. Thus, observables not commuting with some conserved quantity cannot be measured exactly and only approximate measurement is possible. This trade-off in measurement forms the basis of the well-known Wigner-Araki-Yanase theorem. In their study of quantum measurement theory, Wigner and Yanase introduced a quantity called the skew information. As shown in [2], the skew information is essentially a form of Fisher information.

The skew information for a mixed state ρ relative to a self-adjoint "observable", A, is defined as $I(\rho, A) = -\frac{1}{2}\mathrm{Tr}\rho^{1/2}, A]^2$. This definition was subsequently generalized by Dyson as $I_\alpha(\rho, A) = -\frac{1}{2}\mathrm{Tr}([\rho^\alpha, A][\rho^{1-\alpha}, A])$, where $0 < \alpha < 1$ [3]. When $\alpha = 1/2$, $I_\alpha(\rho, X)$ is reduced to the skew information. The convexity of $I_\alpha(\rho, A)$ was finally resolved by Lieb[4, 5].

The von Neumann entropy of ρ, defined as $S(\rho) = -tr\rho \ln \rho$, has been widely used as a measure of the uncertainty of a mixed state. This quantity, profoundly rooted in quantum statistical mechanics, possesses several remarkable and satisfactory properties. Like all measures, the von Neumann entropy, together with its classical analog called the Shannon entropy, is not always the best measure under certain contexts. In [6, 7, 2, 8], the skew information was proposed as means to unify the study of Heisenberg uncertainty relation for mixed states.

It is well know in the standard textbooks that the Heisenberg uncertainty relation for any two self-adjoint operators X and Y is given by

$$V(\rho, X)V(\rho, Y) \geq \frac{1}{4}||\mathrm{Tr}(\rho[X, Y]||^2. \tag{1}$$

Note that [,] is commutator, i.e. $[A, B] = AB - BA$ and the variance of the observable X with respect to ρ is

$$V(\rho, X) = \mathrm{Tr}(\rho X^2) - (\mathrm{Tr}(\rho X))^2. \tag{2}$$

A similar definition applies to $V(\rho, Y)$.

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When ρ is a mixed state, Luo showed that the variance comprises of two terms: a quantum uncertainty term and a classical uncertainty term[6, 7]. He separated the variance into its quantum and classical part by using the skew information. He interpreted $I(\rho,\, X)$ as the quantum uncertainty of X in ρ by the Bohr complementary principle and $V(\rho, X) - I(\rho,\, X)$ as the classical uncertainty of the mixed state. He then considered $U(\rho, X) = \sqrt{V^2(\rho, X) - [V(\rho, X) - I(\rho, X)]^2}$ as a measure of quantum uncertainty. Thus, he obtained the following two inequalities for the uncertainty relation.

$$I(\rho, X)J(\rho, Y) \geq \frac{1}{4}||\mathrm{Tr}(\rho[X, Y]||^2. \tag{3}$$

$$U(\rho, X)U(\rho, Y) \geq \frac{1}{4}||\mathrm{Tr}(\rho[X, Y]||^2. \tag{4}$$

where $J(\rho,\, Y) = \frac{1}{2}\mathrm{Tr}\{\rho^{1/2},\, Y_0\}^2$, and $Y_0 = Y - \mathrm{Tr}(\rho Y)$. The notation $\{\ \}$ is the anticommutator, i.e. $\{A,\, B\} = AB + BA$.

This article is organized as follows: In section 2, we discuss various properties of the Wigner-Yanase-Dyson information. We show using a counter example that it need not satisfy the uncertainty relation obtained from the skew information. In section 3, we formulate an uncertainty relation for Wigner-Yanase-Dyson information. Finally, in section 4, we reiterate our main results. We have also provided two appendices concerning the proof of the new uncertainty principle and additivity of the Wigner-Yanase-Dyson information.

2 Wigner-Yanase-Dyson information violates Heisenberg uncertainty relation

In this paper, we extend the above discussion to Wigner-Yanase-Dyson information. The skew information proposed by Dyson can also be written as

$$\begin{aligned} I_\alpha(\rho, X) &= \mathrm{Tr}(\rho X^2) - \mathrm{Tr}(\rho^\alpha X \rho^{1-\alpha} X) \\ &= \mathrm{Tr}(\rho X_0^2) - \mathrm{Tr}(\rho^\alpha X_0 \rho^{1-\alpha} X_0), \end{aligned} \tag{5}$$

where $X_0 = X - \mathrm{Tr}(\rho X)$. $I_\alpha(\rho, X)$ is positive from Eq. (A5). Similarly, we define $J_\alpha(\rho, Y) = \frac{1}{2}tr(\{\rho^\alpha,\, Y_0\}\{\rho^{1-\alpha},\, Y_0\})$. When $\alpha = 1/2$, $J_\alpha(\rho,\, Y)$ is reduced to $J(\rho,\, Y)$. As well, we can define $J_\alpha(\rho, X)$, $J_\alpha(\rho, A)$, and $J_\alpha(\rho, B)$. By calculating,

$$\begin{aligned} &J_\alpha(\rho, Y) = \\ &\mathrm{Tr}(\rho Y_0^2) + \mathrm{Tr}(\rho^\alpha Y_0 \rho^{1-\alpha} Y_0) = \\ &\mathrm{Tr}(\rho Y^2) + \mathrm{Tr}(\rho^\alpha Y \rho^{1-\alpha} Y) - 2(\mathrm{Tr}\rho Y)^2. \end{aligned} \tag{6}$$

$J_\alpha(\rho,\, Y)$ is also positive from Eq. (A9) in this paper.

Adopting the Luo's interpretations, by the following properties of Wigner-Yanase-Dyson information we interpret $I_\alpha(\rho, X)$ as quantum uncertainty of X in ρ, $V(\rho, X) - I_\alpha(\rho,\, X)$ as the classical mixing uncertainty, and $U_\alpha(\rho, X) = \sqrt{V^2(\rho, X) - [V(\rho, X) - I_\alpha(\rho, X)]^2}$ as a measure of quantum uncertainty. Lieb studied the properties of Wigner-Yanase-Dyson information in [4]. Wigner-Yanase-Dyson information satisfies the following requirements.

(1). Wigner-Yanase-Dyson conjecture about the convexity of $I_\alpha(\rho, X)$ with respect to ρ was proved by Lieb [4].

(2). Wigner-Yanase-Dyson information $I_\alpha(\rho, X)$ is additive under the following sense (See [2] and [4]). Let ρ_1 and ρ_2 be two density operators of two subsystems, and A_1 (resp. A_2) be a self-adjoint operator on H^1 (resp. H^2). $I_\alpha(\rho, X)$ is additive if $I_\alpha(\rho_1 \otimes \rho_2,\, A_1 \otimes I_2 + I_1 \otimes A_2) = I_\alpha(\rho_1,\, A_1) + I_\alpha(\rho_2,\, A_2)$, where I_1 and I_2 are the identity operators for the first and second systems, respectively. For the proof see Appendix B.

(3). $J_\alpha(\rho,\, Y)$ is also additive under the above sense. For the proof see Appendix B.

(4). However, Hansen showed that Wigner-Yanase-Dyson information is not subadditive [11]. For the definition of subadditivity see [4] and [11].

(5). $J_\alpha(\rho, Y)$ is concave with respect to ρ. This is because $tr(\rho Y_0^2)$ is linear operator with respect to ρ and $tr(\rho^\alpha Y_0 \rho^{1-\alpha} Y_0)$ is concave with respect to ρ.

(6). When ρ is pure, $V(\rho, X) = I_\alpha(\rho, X)$. Thus, Wigner-Yahase-Dyson information reduces to the variance. That is, the variance $V(\rho, X)$ does not include the classical mixing uncertainty because of no mixing. In other words, the variance only includes the quantum uncertainty of X in ρ. The case in which $\alpha = 1/2$ was discussed in [7].

The above fact can be argued as follows. When ρ is pure, $tr(\rho^\alpha X_0 \rho^{1-\alpha} X_0) = (tr(\rho X_0))^2 = 0$. Thus, $I_\alpha(\rho, X) = tr(\rho X_0^2) = V(\rho, X)$.

(7). When ρ is a mixed state, $V(\rho, X) \geq I_\alpha(\rho, X)$. This is because $tr(\rho^\alpha X \rho^{1-\alpha} X) = tr((\rho^{\alpha/2} X \rho^{(1-\alpha)/2})$ $(\rho^{\alpha/2} X \rho^{(1-\alpha)/2})^\dagger) \geq 0$. Also, see Eq. (A3) in this paper. The case in which $\alpha = 1/2$ was discussed in [7].

(8). When ρ and A commute, according to the discussion for the skew information in [6, 8], the quantum uncertainty should vanish and thus, the variance only includes the classical uncertainty. We can argue that the above conclusion is also true for Wigner-Yanase-Dyson information. When ρ and A commute, it is well known that ρ and A have the same orthonormal eigenvector basis [9]. Hence, ρ^α and A also commute. By the definition in Eq. (5), Wigner-Yanase-Dyson information $I_\alpha(\rho, X)$ vanishes.

However, $I_\alpha(\rho, X)$ and $J_\alpha(\rho, Y)$ do not satisfy Eq. (3). We give the following counter example for Eq. (3).

Let $n = 2$, $\alpha = 1/4$, and ρ have the eigenvalues $\lambda_1 = 1/4$ and $\lambda_2 = 3/4$. Since A and B are self-adjoint, then we write $A = \begin{pmatrix} x & u+iv \\ u-iv & y \end{pmatrix}$, $B = \begin{pmatrix} a & c+di \\ c-di & b \end{pmatrix}$. In this example, $u = 4$, $v = 2$, $a = b = 0$, $c = 1$, and $d = -5$. By calculating $I_\alpha(\rho, A)$ in Eq. (A5) and $J_\alpha(\rho, B)$ in Eq. (A8), $I_\alpha(\rho, A) J_\alpha(\rho, B) = [1 - (\lambda_1^\alpha \lambda_2^{1-\alpha} + \lambda_2^\alpha \lambda_1^{1-\alpha})^2](u^2 + v^2)(c^2 + d^2) = 99.83$. By calculating $\mathrm{Tr}(\rho[A, B]$ in Eq. (A11), $\frac{1}{4}|\mathrm{Tr}(\rho[A, B]|^2 = (\lambda_1 - \lambda_2)^2(cv - du)^2 = 121$. Hence, it violates Eq. (3). It implies that the bound on the right side of the inequality in Eq. (3) is too large in this example. We need to get the appropriate lower bound for Wigner-Yanase-Dyson information, i.e., we need to modify the term on RHS of the inequality.

3 The general uncertainty relation

We replace $\mathrm{Tr}(\rho[X, Y]$ with $l_\alpha(\rho, X, Y)$ which is defined as follows:

$$l_\alpha(\rho, X, Y) = \mathrm{Tr}(\rho[X, Y]) - \mathrm{Tr}\rho^{|2\alpha - 1|}[X, Y]. \tag{7}$$

When $\alpha = 1/2$, $l_\alpha(\rho, X, Y)$ reduces to $\mathrm{Tr}(\rho[X, Y])$. In [6], Luo defined $k = \mathrm{i}[\rho^{1/2}, X_0]t + \{\rho^{1/2}, Y_0\}$, where $t \in R$ and i is an imaginary number. From $\mathrm{Tr}(kk^\dagger) \geq 0$, by expanding $\mathrm{Tr}(kk^\dagger)$, he derived $\mathrm{Tr}(kk^\dagger) = 2(I[\rho, X]t^2 + \mathrm{i}(tr(\rho[X, Y])t + J[\rho, Y]) \geq 0$. Since the above inequality is true for any real t, Luo obtained the inequality in Eq. (3). However, unlike his previous case, the form of $I_\alpha(\rho, X)$ does not allow us to employ the trick $k = \mathrm{i}[\rho^\alpha, X_0]t + \{\rho^\alpha, Y_0\}$ nor $k = \mathrm{i}[\rho^{1-\alpha}, X_0]t + \{\rho^{1-\alpha}, Y_0\}$ to derive the uncertainty relation from $\mathrm{Tr}(kk^\dagger) \geq 0$. The proof becomes more involved and one needs to modify the RHS of the previous uncertainty relation.

In Appendix A, we see that if A and B are self-adjoint observables, then

$$I_\alpha(\rho, A) J_\alpha(\rho, B) \geq \frac{1}{4}||l_\alpha(\rho, A, B)||^2, \tag{8}$$

and

$$I_\alpha(\rho, B) J_\alpha(\rho, A) \geq \frac{1}{4}||l_\alpha(\rho, A, B)||^2. \tag{9}$$

If we denote $U_\alpha(\rho, \mathcal{O})$ as $\sqrt{V^2(\rho, \mathcal{O}) - [V(\rho, \mathcal{O}) - I_\alpha(\rho, \mathcal{O})]^2}$, we see that by Eq. (2) and Eq.(5) (and the analogous form for $J_\alpha(\rho, \mathcal{O})$), $U_\alpha(\rho, \mathcal{O}) = \sqrt{I_\alpha(\rho, \mathcal{O}) J_\alpha(\rho, \mathcal{O})}$, where $\mathcal{O}$ is either the operator A or B. Thus, we obtain our main result from Eqs. (8) and (9),

$$U_\alpha(\rho, A) U_\alpha(\rho, B) \geq \frac{1}{4}||l_\alpha(\rho, X, Y)||^2. \tag{10}$$

For the counter example in Sec. 2, a direct calculation of Eq. (A13) yields $\frac{1}{4}||l_\alpha(\rho, A, B)||^2 = 8.6874$. Therefore, the inequality in Eq. (8) holds in this case.

4 Summary

In [6], Luo presented a refined Heisenberg uncertainty relation. In this paper, we demonstrate some properties of Wigner-Yanase-Dyson information and provide a counter example to show that Wigner-Yanase-Dyson information does not in general satisfy Heisenberg uncertainty relation. We have also proposed a new general uncertainty relation of mixed states based on Wigner-Yanase-Dyson information. Bell-type inequalities based on the skew information have been proposed as nonlinear entanglement witnesses [12]. We note here that similar Bell-type inequalities with the advantage of an additional α parameter for fine adjustments could also be constructed from the uncertainty principle derived from the Wigner-Yanase-Dyson information.

Appendix A. Proof of uncertainty relation

By spectral decomposition, there exists an orthonormal basis $\{x_1,..., x_n\}$ consisting of eigenvectors of ρ. Let λ_1, ..., λ_n be the corresponding eigenvalues, where $\lambda_1 + ... + \lambda_n = 1$ and $\lambda_i \geq 0$. Thus, ρ has a spectral representation

$$\rho = \lambda_1|x_1\rangle\langle x_1| + ... + \lambda_n|x_n\rangle\langle x_n|. \tag{A1}$$

1. Calculating $I_\alpha(\rho,\ A)$

By Eq. (A1), $\rho A^2 = \lambda_1|x_1\rangle\langle x_1|A^2 + ... + \lambda_n|x_n\rangle\langle x_n|A^2$ and

$$\begin{aligned} \mathrm{Tr}\rho A^2 &= \lambda_1\langle x_1|A^2|x_1\rangle + ... + \lambda_n\langle x_n|A^2|x_n\rangle \\ &= \lambda_1||A|x_1||^2 + ... + \lambda_n||A|x_n||^2. \end{aligned} \tag{A2}$$

Moreover, since $\rho^\alpha A = \lambda_1^\alpha|x_1\rangle\langle x_1|A + ... + \lambda_n^\alpha|x_n\rangle\langle x_n|A$ and $\rho^{1-\alpha}A = \lambda_1^{1-\alpha}|x_1\rangle\langle x_1|A + ... + \lambda_n^{1-\alpha}|x_n\rangle\langle x_n|A$, we have, $\rho^\alpha A\rho^{1-\alpha}A = \sum_{i,j=1}\lambda_i^\alpha\lambda_j^{1-\alpha}|x_i\rangle\langle x_i|A|x_j\rangle\langle x_j|A$. Thus

$$\begin{aligned} \mathrm{Tr}\rho^\alpha A\rho^{1-\alpha}A &= \sum_{i,j=1}\lambda_i^\alpha\lambda_j^{1-\alpha}\langle x_i|A|x_j\rangle\langle x_j|A|x_i\rangle \\ &= \sum_{i,j=1}\lambda_i^\alpha\lambda_j^{1-\alpha}||\langle x_i|A|x_j\rangle||^2. \end{aligned} \tag{A3}$$

From Eqs. (5), (A2) and (A3),

$$I_\alpha(\rho, A) = \sum_{i=1}\lambda_i||A|x_i||^2 - \sum_{i,j=1}\lambda_i^\alpha\lambda_j^{1-\alpha}||\langle x_i|A|x_j\rangle||^2. \tag{A4}$$

Let $A = \{A_{ij}\}$ (resp. $B = \{B_{ij}\}$) be the matrix representation of the operator A (resp. B) corresponding to the orthonormal basis $\{x_1,..., x_n\}$. Then $\langle x_i|A|x_j\rangle = A_{ij}$, and

$$\begin{aligned} I_\alpha(\rho, A) &= \sum_{i\neq j}(\lambda_i - \lambda_i^\alpha\lambda_j^{1-\alpha})\,||A_{ij}||^2 \\ &= \sum_{i<j}(\lambda_i + \lambda_j - \lambda_i^\alpha\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^\alpha)\,||A_{ij}||^2\,. \end{aligned} \tag{A5}$$

2. ***Calculating*** $J_\alpha(\rho, B)$

Similarly, from Eqs. (6) and (A1), we can obtain

$$\begin{aligned} J_\alpha(\rho, B) &= \sum_{i=1} \lambda_i ||B|x_i||^2 + \sum_{i,j=1} \lambda_i^\alpha \lambda_j^{1-\alpha} ||\langle x_i|B|x_j\rangle||^2 \\ &\quad -2(\sum \lambda_i \langle x_i|B|x_i\rangle)^2. \end{aligned} \tag{A6}$$

Let $\langle x_i|B|x_j\rangle = B_{ij}$. Then, from Eq. (A6),

$$\begin{aligned} J_\alpha(\rho, B) &= 2\sum_{i=1} \lambda_i \left|B_{ii}\right|^2 - 2(\sum_{i=1} \lambda_i B_{ii})^2 \\ &\quad + \sum_{i\neq j} (\lambda_i + \lambda_i^\alpha \lambda_j^{1-\alpha}) \left||B_{ij}|\right|^2. \end{aligned} \tag{A7}$$

By simplifying,

$$\begin{aligned} J_\alpha(\rho, B) &= 2\sum_{i=1} \lambda_i \left|B_{ii}\right|^2 - 2(\sum_{i=1} \lambda_i B_{ii})^2 \\ &\quad + \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha) \left||B_{ij}|\right|^2. \end{aligned} \tag{A8}$$

Since x^2 is convex, $(\sum_{i=1} \lambda_i B_{ii})^2 \leq \sum_{i=1} \lambda_i \left|B_{ii}\right|^2$. So from Eq. (A8),

$$J_\alpha(\rho, B) \geq \sum_{i<j} (\lambda_i + \lambda_j + \lambda_i^\alpha \lambda_j^{1-\alpha} + \lambda_i^{1-\alpha}\lambda_j^\alpha) \left||B_{ij}|\right|^2. \tag{A9}$$

3. ***Calculating*** $l_\alpha(\rho,\ A,\ B)$

First we calculate $\mathrm{Tr}(\rho[A,\ B])$. By Eq. (A1), $\rho[A,B] = \lambda_1|x_1\rangle\langle x_1|[A,\ B] + ... + \lambda_n|x_n\rangle\langle x_n|[A,\ B]$ and $\mathrm{Tr}(\rho[A,\ B]) = \lambda_1\langle x_1|[A,\ B]|x_1\rangle + ... + \lambda_n\langle x_n|[A,\ B]|x_n\rangle$. It is well known that $Re\langle x_i|[A,\ B]|x_i\rangle = 0$ and $\langle x_i|[A,\ B]|x_i\rangle = \mathrm{i}(2Im\langle x_i|AB|x_i\rangle)$, where i is an imaginary number. Consequently, $\mathrm{Tr}(\rho[A,\ B]) = 2\mathrm{i}(\lambda_1 Im\langle x_1|AB|x_1\rangle + ... + \lambda_n Im\langle x_n|AB|x_n\rangle)$. Therefore we obtain

$$\begin{aligned} \mathrm{Tr}(\rho[A,B]) &= 2\mathrm{i}Im(\lambda_1\langle x_1|AB|x_1\rangle + ... + \lambda_n\langle x_n|AB|x_n\rangle) \\ &= 2\mathrm{i}Im\sum_{j\neq i} \lambda_i A_{ij} B_{ji}. \end{aligned} \tag{A10}$$

Note that in Eq. (A10) A_{ii} and B_{ii} are real because A and B are self-adjoint. Since $A_{ij}B_{ji} = (A_{ji}B_{ij})^*$, $\Im \sum_{j\neq i} \lambda_i A_{ij} B_{ji} = Im \sum_{i<j} (\lambda_i - \lambda_j) A_{ij} B_{ji}$. Thus, by simplifying,

$$\mathrm{Tr}(\rho[A,B]) = 2\mathrm{i}Im\sum_{i<j} (\lambda_i - \lambda_j) A_{ij} B_{ji}. \tag{A11}$$

Moreover,

$$\mathrm{Tr}\rho^{|2\alpha-1|}[A,B] = 2\mathrm{i}Im\sum_{i<j} (\lambda_i^{|2\alpha-1|} - \lambda_j^{|2\alpha-1|}) A_{ij} B_{ji}. \tag{A12}$$

Hence, from Eqs. (7), (A11) and (A12),

$$l_\alpha(\rho, A, B) = 2\mathrm{i}\sum_{i<j} (\lambda_i - \lambda_j - (\lambda_i^{|2\alpha-1|} - \lambda_j^{|2\alpha-1|})) Im(A_{ij} B_{ji}). \tag{A13}$$

4. *The proof of the uncertainty relation*

From Eqs. (A5), (A9) and (A13), for Eq. (8) we need to show

$$\begin{aligned} & [\sum_{i<j}(\lambda_i+\lambda_j-\lambda_i^{\alpha}\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha}\lambda_j^{\alpha})\,||A_{ij}||^2][\sum_{i<j}(\lambda_i+\lambda_j+\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})\,||B_{ij}||^2] \\ \geq \quad & \{\sum_{i<j}[\lambda_i-\lambda_j-(\lambda_i^{|2\alpha-1|}-\lambda_j^{|2\alpha-1|})]Im(A_{ij}B_{ji})\}^2. \end{aligned} \tag{A14}$$

It is easy to know $[Im(A_{ij}B_{ji})]^2 \leq ||A_{ij}||^2\,||B_{ij}||^2$. Note that $\lambda_i+\lambda_j-\lambda_i^{\alpha}\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha}\lambda_j^{\alpha}=(\lambda_i^{\alpha}-\lambda_j^{\alpha})(\lambda_i^{1-\alpha}-\lambda_j^{1-\alpha})\geq 0$. By the Cauchy-Schwartz inequality, the LHS of the inequality in Eq. (A14) $\geq \{\sum[(\lambda_i+\lambda_j)^2-(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})^2]^{1/2}Im(A_{ij}B_{ji})\}^2$. Finally, what needs to be shown is

$$\begin{aligned} & (\lambda_i+\lambda_j)^2-(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})^2 \\ \geq \quad & |(\lambda_i-\lambda_j)-(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})|^2. \end{aligned} \tag{A15}$$

It is easy to see that

$$\begin{aligned} & (\lambda_i+\lambda_j)^2-(\lambda_i^{\alpha}\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^{\alpha})^2 \\ = \quad & (\lambda_i-\lambda_j)^2-(\lambda_i^{\alpha}\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha}\lambda_j^{\alpha})^2. \end{aligned}$$

When $\alpha \geq 1/2$,

$$\begin{aligned} & (\lambda_i-\lambda_j)^2-(\lambda_i^{\alpha}\lambda_j^{1-\alpha}-\lambda_i^{1-\alpha}\lambda_j^{\alpha})^2 \\ = \quad & (\lambda_i-\lambda_j)^2-\lambda_i^{2(1-\alpha)}\lambda_j^{2(1-\alpha)}(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})^2 \\ \geq \quad & (\lambda_i-\lambda_j)^2-(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})^2 \\ = \quad & |(\lambda_i-\lambda_j)-(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})| \\ & \times\left|(\lambda_i-\lambda_j)+(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})\right| \\ \geq \quad & |(\lambda_i-\lambda_j)-(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})|^2. \end{aligned}$$

Note that the last inequality holds because $(\lambda_i-\lambda_j)$ and $(\lambda_i^{2\alpha-1}-\lambda_j^{2\alpha-1})$ have the same sign. Also, when $0<\alpha\leq 1/2$, we can prove the inequality in Eq. (A15) as follows: Let $\beta=1-\alpha$ with $1/2\leq\beta<1$. Replacing α in Eq. (A15) with $1-\ \beta$, we obtain $(\lambda_i+\lambda_j)^2-(\lambda_i^{1-\beta}\lambda_j^{\beta}+\lambda_i^{\beta}\lambda_j^{1-\beta})^2\geq|(\lambda_i-\lambda_j)-(\lambda_i^{2\beta-1}-\lambda_j^{2\beta-1})|^2$. This ends the proof.

Appendix B. Additivity

The quantity $J_\alpha(\rho,\,B)$ is additive in the following sense: $J_\alpha(\rho_1\otimes\rho_2,\,B_1\otimes I_2+I_1\otimes B_2)=J_\alpha(\rho_1,\,B_1)+J_\alpha(\rho_2,\,B_2)$. Using the notation in [4], the proof proceeds by letting $\rho_{12}=\rho_1\otimes\rho_2$ and $L=B_1\otimes I_2+I_1\otimes B_2$. Setting $\rho_{12}^{\alpha}=\rho_1^{\alpha}\otimes\rho_2^{\alpha}$, we have
$\rho_{12}^{\alpha}L\rho_{12}^{1-\alpha}L=\rho_1^{\alpha}B_1\rho_1^{1-\alpha}B_1\otimes\rho_2+\rho_1^{\alpha}B_1\rho_1^{1-\alpha}\otimes\rho_2B_2+\rho_1B_1\otimes\rho_2^{\alpha}B_2\rho_2^{1-\alpha}+\rho\otimes\rho_2^{\alpha}B_2\rho_2^{1-\alpha}B_2$, and

$$\begin{aligned} & \mathrm{Tr}(\rho_{12}^{\alpha}L\rho_{12}^{1-\alpha}L) \\ = \quad & \mathrm{Tr}(\rho_1^{\alpha}B_1\rho_1^{1-\alpha}B_1)+2\mathrm{Tr}(\rho_1B_1)\mathrm{Tr}(\rho_2B_2)+\mathrm{Tr}(\rho_2^{\alpha}B_2\rho_2^{1-\alpha}B_2). \end{aligned} \tag{B1}$$

Similarly,

$$\mathrm{Tr}(\rho_{12}^{\alpha}L^2)=\mathrm{Tr}(\rho_1B_1^2)+2\mathrm{Tr}(\rho_1B_1)\mathrm{Tr}(\rho_2B_2)+\mathrm{Tr}(\rho_2B_2^2). \tag{B2}$$

From the above Eqs. (B1) and (B2), we can derive $I_\alpha(\rho,\,B)$ is additive.

Similarly,

$$\mathrm{Tr}(\rho_{12}^{\alpha}L)=\mathrm{Tr}(\rho_1 B_1)+\mathrm{Tr}(\rho_2 B_2). \tag{B3}$$

By Eqs. (B1), (B2), and (B3), and the definition of $J_\alpha(\rho,\, B)$ in Eq. (6), we can conclude that $J_\alpha(\rho,\, B)$ is additive.

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References

[1] Fisher information has been discussed extensively in literature on statistical estimation theory. A particularly insightful survey of the connection between Fisher information and the Heisenberg uncertainty principle can be found in arXiv: quant-ph//0309184

[2] S. Luo, Phys. Rev. Lett., **91**, 180403 (2003).

[3] E.P. Wigner and M. M. Yanase, Proc. Nat. Acad. Sci. U.S.A. **49**, 910-918 (1963).

[4] E.H. Lieb, Adv. Math. **11**, 267 (1973).

[5] E.H. Lieb and M.M. Ruskai, Phys. Rev. Lett. **30**, 434 (1973).

[6] S. Luo, Phys. Rev. A **72**, 042110 (2005).

[7] S. Luo, Phys. Rev. A **73**, 022324 (2006).

[8] S. Luo, Theor. Math. Phys. **143**, 681 (2005).

[9] M. Hirvensalo, Quantum Computing, Springer-Verlag, Berlin, (2001).

[10] M. A. Nielsen and I. L. Chuang, see p.89, Quantum Computation and Quantum Information, Cambridge Univ. Press, Cambridge (2000).

[11] F. Hansen, J. Stat. Phys., **126**, 643 (2007).

[12] Z. Chen, Phys. Rev. A **71**, 052302 (2005).